International Journal of Theoretical Physics, Vol. 30, No. 4, 1991

Associated Symplectic and Co-symplectic Structures

F. A. M. Frescura[1] and G. Lubczonok[2]

Received June 26, 1990

A co-symplectic structure on the cotangent bundle T^*X of an arbitrary manifold X is defined, and the notion of associated symplectic and co-symplectic structures is introduced. By way of example, the two-dimensional case is considered in some detail. The general case is investigated, and some implications of these results for polarizations in geometric quantization are considered.

1. INTRODUCTION

In a recent article (Frescura and Lubczonok, 1990), we introduced a new geometric structure which we proposed to call *co-symplectic geometry*. This structure is based on a symmetric bilinear form of signature zero and leads to a geometry that is, in many respects, analogous to the symplectic geometry. Its usefulness lies principally in the fact that it provides scope for the geometrization of a number of familiar structures in physics which are not so easily amenable by the methods of symplectic geometry. These include the angular momentum operators of quantum theory, the Dirac operators in relativistic quantum mechanics, and the fermionic creation and annihilation operators of quantum field theory. It is anticipated that, in conjunction with the more familiar symplectic geometry, the co-symplectic geometry will go some way to providing the tools necessary for a full geometrization of physics.

In our previous article, we investigated the geometry of co-symplectic vector spaces. Effectively, therefore, we have studied only the local properties of the co-symplectic geometry. It is well known, however, that even the most rudimentary physical systems give rise to nontrivial global structures that strongly influence their properties, stability, and long-term behavior.

[1]Department of Physics and Electronics, Rhodes University, P.O. Box 94, Grahamstown, 6140 South Africa.
[2]Department of Mathematics and Applied Mathematics, Rhodes University, P.O. Box 94, Grahamstown, 6140 South Africa.

0020-7748/91/0400-0567$06.50/0

Any credible geometrization will therefore have to be based, not on the trivial local geometry examined thus far, but on some nontrivial globalization of it. Accordingly, we propose in this paper to extend the ideas contained in our previous article, and to examine some aspects of the geometry of co-symplectic manifolds.

The natural setting for mechanical systems in both classical and quantum physics is not the configuration space, but the phase space, or some extension of it. The co-symplectic structures that are principally of interest to us are thus those that can be introduced into the cotangent bundle of some suitably chosen manifold, or some construct on the cotangent bundle. In this paper, therefore, we concentrate mainly on these.

The coexistence of spin and Hamiltonian structures in quantum theory and in quantum field theory (Berezin, 1966, 1987), and also in some recently proposed classical models (Berezin and Marinov, 1977; Gomis *et al.*, 1985; Sherry, 1989), leads us to examine the interrelationship of symplectic and co-symplectic geometries. In the theory presented in this paper, we accomplish this by introducing into the same cotangent bundle simultaneously both symplectic and co-symplectic structure. This leads to the notion of *associated co-symplectic structures*. From a different point of view, the co-symplectic structure can be interpreted as a special kind of Riemannian structure. The compatibility problem then coincides with the problem of introducing a natural connection into the symplectic geometry relative to which the symplectic structure becomes covariantly constant. The conditions under which this can be done are derived.

The model on which initially, we base our development is that of the symplectic geometry. We begin, therefore, in Section 2, with a brief review of some important aspects of that structure. This enables us to review a construction that we apply in Section 3 to the co-symplectic case. In Section 3 we define a co-symplectic structure on the cotangent bundle T^*X of an arbitrary manifold X. This is done by appeal to a partition of unity. We offer also an alternative interpretation of this construction at the end of the section in terms of Hessians of local phase functions for waves on the configuration space. In Section 4, we define associated symplectic and co-symplectic structures and show how a new co-symplectic structure can be obtained from a given one and an arbitrary Riemannian structure on the configuration space. In Section 5 we show how real symplectic and co-symplectic structures both arise naturally in the context of complex symplectic manifolds.

We examine in detail associated symplectic and co-symplectic structures, first the two-dimensional case in Section 6, and then the general case in Section 7. Some implications of these results for polarizations in geometric quantization are considered in Section 8.

2. SYMPLECTIC MANIFOLDS

To set up the theory of co-symplectic manifolds, we shall imitate the methods of symplectic geometry. It is convenient therefore to review first some of the basic features of symplectic geometry. This will also give us opportunity to establish our notation.

Let M be a manifold of even dimension, and set $n = \frac{1}{2}\dim M$. A *symplectic structure* on M is a two-form $\omega \in \Lambda^2(M)$ that is nondegenerate and closed. Thus,

$$v \lrcorner \omega = 0 \Leftrightarrow v = 0 \quad \forall v \in TM$$

and

$$d\omega = 0$$

The two-form ω is called a *symplectic form*, and the pair (M, ω) is called a *symplectic manifold.*

The fundamental theorem of symplectic geometry[3] is the theorem of Darboux, which allows us to introduce local canonical coordinates $\{q^i, p_i\}$, $i = 1, \ldots, n$, in which

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i \tag{1}$$

Thus, a local chart can be found around each point m of M in which ω takes its canonical form throughout the charted neighborhood. This means that any symplectic form is equivalent locally to the standard symplectic form

$$\Omega = \sum_{i=1}^{n} dp_i \wedge dq^i$$

on $T^*\mathfrak{R}^n \approx \mathfrak{R}^n \times (\mathfrak{R}^n)^*$. In this sense, every symplectic form can be said to be *covariantly constant* and every symplectic manifold can be said to be *locally flat.*

Now let X be any manifold of dimension n. Then the cotangent bundle T^*X over X admits a symplectic structure in a natural way. The naturalness of this structure is emphasized by the fact that it can be defined globally by a standard procedure without recourse to local coordinates. This is the method followed in Abraham and Marsden (1978, pp. 178–179). It is more useful for our purposes, however, to consider how this construction might be carried out in terms of local coordinate charts.

[3]Excellent treatments of symplectic geometry can be found in Guillemin and Sternberg (1977, 1984), Abraham and Marsden (1978), Woodhouse (1980), and Liebermann and Marle (1987). We have made extensive use of these works in our treatment here.

Let $\mathscr{A}$ be an atlas on X, and $(U, \{x^i\})$ a local coordinate chart. A local coordinate chart $\{x^i, \xi_i\}$ on $T^*U \subset T^*X$ can be constructed in a natural way (Abraham and Marsden, 1978, pp. 46–47). Coordinate charts of this kind are called *natural coordinates.* Now let $(V, \{y^i\})$ be another local chart on X such that $U \cap V \neq \varnothing$, and let the natural extension of $(V, \{y^i\})$ to a chart of $T^*V \subset T^*X$ have coordinates $\{y^i, \eta_i\}$. The charts U and V are related on the region $U \cap V$ of overlap by a local coordinate transformation $\phi_{VU} = \phi_V \circ \phi_U^{-1}|_{\phi_U(U \cap V)}$ which is of the form

$$y^i = y^i(x), \qquad i, j = 1, \ldots, n \tag{2}$$

This transformation in turn will induce a local change of coordinates on T^*X, given by

$$y^i = y^i(x), \qquad \xi_i = \eta_j \frac{\partial x^j}{\partial y^i} \tag{3}$$

Denote transformation (3) by $\tilde{\phi}$. Then in the region $U \cap V$ of overlap of the coordinates, the coordinate basis of $T(T^*U)$ transforms under $\tilde{\phi}$ according to

$$\tilde{\phi}_*: \quad \frac{\partial}{\partial x^i} \mapsto \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} + \xi_k \frac{\partial^2 x^k}{\partial y^l \partial y^j} \frac{\partial y^l}{\partial x^i} \frac{\partial}{\partial \eta_j} \tag{4}$$

$$\tilde{\phi}_*: \quad \frac{\partial}{\partial \xi_i} \mapsto \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial \eta_j} \tag{5}$$

Similarly, the coordinate basis of $T^*(T^*V)$ transforms according to

$$\tilde{\phi}^*: \quad dy^i \mapsto \frac{\partial y^i}{\partial x^j} dx^j \tag{6}$$

$$\tilde{\phi}^*: \quad d\eta_i \mapsto \xi_k \frac{\partial^2 x^k}{\partial y^l \partial y^i} \frac{\partial y^l}{\partial x^j} dx^j + \frac{\partial x^j}{\partial y^i} d\xi_j \tag{7}$$

The Jacobian of this transformation, in block diagonal form, is thus

$$\mathscr{J} = \begin{pmatrix} \dfrac{\partial y^i}{\partial x^j} & 0 \\ \xi_k \dfrac{\partial^2 x^k}{\partial y^l \partial y^j} \dfrac{\partial y^l}{\partial x^i} & \left(\left[\dfrac{\partial y^i}{\partial x^j} \right]^{-1} \right)^T \end{pmatrix} \tag{8}$$

We now introduce the local symplectic form

$$\omega^U = \sum_{i=1}^{n} d\xi_i \wedge dx^i$$

in each local chart U of the atlas $\mathscr{A}$. Then, on $U \cap V$, the local forms ω^U and ω^V will coincide,

$$\omega^U|_{U \cap V} = \omega^V|_{U \cap V} \tag{9}$$

This property is guaranteed by the symmetry of the partial derivatives

$$\frac{\partial^2 x^k}{\partial y^i \, \partial y^j}$$

and is the feature that allows us to extend the local symplectic structure in a natural way to cover the entire manifold. We can therefore define the natural symplectic structure ω on T^*X by requiring that in each local chart U of the atlas $\mathscr{A}$ we have

$$\omega|_U = \omega^U \tag{10}$$

a demand that is consistent by virtue of (9).

3. CO-SYMPLECTIC STRUCTURE ON T^*X

We now apply a construction analogous to the one described in Section 2, to the co-symplectic case. Let $\mathscr{A}$ be an atlas on X, and $(U, \{x^i\})$ a local coordinate chart that has been extended to a natural coordinate system $\{x^i, \xi_i\}$ of $T^*U \subset T^*X$. Define

$$\sigma^U = \sum_{i=1}^{n} (d\xi_i \otimes dx^i + dx^i \otimes d\xi_i) \tag{11}$$

Thus, σ^U is the local co-symplectic structure in canonical form (Frescura and Lubczonok, 1990). σ^U is evidently a Riemannian tensor of signature zero on T^*U. It has components

$$[(\sigma^U)_{ij}] = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \equiv S \tag{12}$$

Here I_n is the identity $n \times n$ matrix, and S is the standard co-symplectic matrix.

The extension of σ^U to the other local charts in the atlas $\mathscr{A}$ of X does not proceed as smoothly as in the corresponding symplectic case. This is due to the fact that the matrices (8) do not preserve the components of σ^U. We therefore have to use a partition of unity to glue together the local structures $\{\sigma^U\}$ to obtain a global co-symplectic structure on T^*X.

Let X now be a paracompact manifold, and $\{(U, x_U)\}_{U \in \Lambda}$ a locally finite atlas on X. We take a partition of unity $\{\theta_U\}_{U \in \Lambda}$ subject to the covering $\{U\}_{U \in \Lambda}$ of X. Denote the induced atlas on T^*X by $\{(U^*, x_U^*)\}_{U \in \Lambda}$. Given a coordinate chart (U^*, x^*_U), where $x^*_U = \{x^i_U, \xi^U_i\}$, we define the local co-symplectic tensors $\{\sigma^U\}$ by (11). Now put

$$\sigma = \sum_{U \in \Lambda} \theta_U \sigma^U \tag{13}$$

σ is evidently a smooth symmetric tensor on the manifold T^*X.

We shall show that σ is nonsingular and with signature zero. The proof is based on the following simple observation. Let

$$\Omega = \begin{pmatrix} A & 0 \\ B & (A^T)^{-1} \end{pmatrix}$$

where A is any $n \times n$ nonsingular matrix. Then

$$\Omega^T S \Omega = \begin{pmatrix} A^T B + B^T A & I_n \\ I_n & 0 \end{pmatrix} \tag{14}$$

where S is the canonical matrix (12). Now let $x \in V \subset X$, where V is an open neighborhood of x such that only a finite number of local charts $U_1, U_2, \ldots, U_s$ intersects V. Then

$$\sigma|_V = \sum_{j=1}^{s} \theta_{U_j} \sigma^{U_j} \tag{15}$$

We now evaluate the components of σ^{U_j} on the intersection $U_j \cap U_i$ in the coordinates $\{x_{U_i}\}$ of U_i. From (8) and (13) we obtain

$$\sigma^{U_j} = \begin{pmatrix} S_{U_j}(x)\xi & I_n \\ I_n & 0 \end{pmatrix} \tag{16}$$

From (14), with Ω taken to be the Jacobian matrix (8), we see that the matrix $S_{U_j}(x)\xi$ is symmetric. Thus, in every coordinate chart (U^*, x^*), the tensor σ has the form

$$\sigma_U(x, \xi) = \begin{pmatrix} S_U(x)\xi & I_n \\ I_n & 0 \end{pmatrix} \tag{17}$$

where $S_U(x)\xi$ is a symmetric $n \times n$ matrix that depends linearly on ξ. From this it is evident that the tensor σ on T^*X is nonsingular and with signature zero. We have thus proved the following result.

Proposition 1. If X is a paracompact manifold, then on the manifold T^*X there is a co-symplectic structure σ such that, in every chart $\{U^*, x^*_U\}$

induced from a chart $\{U, x_U\}$ on X,

$$\sigma = \begin{pmatrix} S_U(x)\xi & I_n \\ I_n & 0 \end{pmatrix} \tag{18}$$

The above construction of a co-symplectic structure σ on T^*X admits also another interpretation. Given a coordinate chart $\{x^i, \xi_i\}$ on T^*X induced from X, we define the function $f: U \times (\mathfrak{R}^n)^* \hookrightarrow T^*X$ by

$$f_U(x, \xi) = \sum_{i=1}^{n} x^i \xi_i$$

Clearly,

$$\text{Hess } f_U(x, \xi) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

Thus, $\sigma(x, \xi)$ arises as a result of glueing together Hessians of local f_U functions. This reinterpretation is not inconsequential. The functions $f_U(x, \xi)$ and their Hessians arise in a natural way in the theory of waves, Fourier transforms, and quantization. They are also associated in a natural way with a co-symplectic structure on cotangent bundles, as shown above, and in this way they play a fundamental role in the application of co-symplectic methods to the aforementioned areas. The development of these ideas will be reported in detail in a later publication.

4. ASSOCIATED SYMPLECTIC AND CO-SYMPLECTIC STRUCTURES

For applications to systems with spin, in which both symplectic and co-symplectic structures occur, it is useful to interrelate symplectic and co-symplectic geometries. With a slight change of interpretation, the co-symplectic geometry can be regarded as a Riemannian geometry on a $2n$-dimensional manifold in which the fundamental tensor has signature zero. Thus, to interrelate symplectic and co-symplectic geometries is equivalent to uniting the symplectic geometry with a particular kind of Riemannian structure. Seen from this point of view, the idea is not entirely new, and should be compared with attempts to introduce a natural connection into symplectic geometry. See, for example, Hess (1980), Cariñena and Ibort (1984), and the references quoted in these papers.

We propose the following terminology.

Definition 1. A Riemannian structure (X, σ), where σ_{ij} is nonsingular, symmetric, and of signature zero, will be called a *Riemannian hyperbolic structure* (Porteous, 1969).

Definition 2. Let (M, ω) be a symplectic structure on a manifold M. We say a Riemannian hyperbolic structure (M, σ) is a *co-symplectic structure associated with* (M, ω) if for each point $x \in M$ there are canonical coordinates (p, q) about x (i.e., $\omega = dp \wedge dq$), in which the tensor σ has the form

$$\sigma = \begin{pmatrix} G & I_n \\ I_n & 0 \end{pmatrix} \tag{19}$$

where G is an appropriate symmetric $n \times n$ matrix. If, moreover,

$$G(q, p) = G(q)p \tag{20}$$

where $G(q)p$ is a linear function of the variables p, then we shall say that (M, σ) is *a co-symplectic structure cotangentially associated with* (M, ω).

Now let X be any manifold. Then the manifold T^*X carries a natural symplectic structure ω. Given any co-symplectic structure σ on T^*X associated with (T^*X, ω), one can generate a new co-symplectic structure $\tilde{\sigma}$ from the pullback $\tilde{g} = \pi^*g$ of any Riemannian structure g on X by the canonical projection $\pi : T^*X \to X$. Put

$$\tilde{\sigma} = \sigma + \pi^*g \tag{21}$$

In a local canonical chart for ω, this yields

$$\tilde{\sigma} = \begin{pmatrix} G+g & I_n \\ I_n & 0 \end{pmatrix} \tag{22}$$

which shows that $\tilde{\sigma}$ is a co-symplectic structure associated with the natural symplectic structure on T^*X. Note that the metric $\tilde{\sigma}$ restricted to the submanifold $X \subset T^*X$ yields g when σ is given by (20), since σ vanishes on X.

In the remainder of this article, we shall explore associated symplectic and co-symplectic structures.

5. NATURAL CO-SYMPLECTIC STRUCTURES ON COMPLEX SYMPLECTIC MANIFOLDS

A co-symplectic structure arises in a natural way on a complex symplectic manifold as follows. Let M be a complex analytic manifold of (complex) dimension $2m$, and let ω be an holomorphic symplectic form on X. The holomorphic structure on M allows us to define

$$\tilde{\omega}(\xi, \eta) = \omega(\xi, \bar{\eta}) \tag{23}$$

where $\bar{\eta}$ is the complex conjugate of η, and $\xi, \eta \in T_xM$. Denote

$$\omega_R(\xi, \eta) = \Re\{\tilde{\omega}(\xi, \eta)\} \tag{24}$$

and

$$\omega_I(\xi, \eta) = \Im\{\tilde{\omega}(\xi, \eta)\} \tag{25}$$

By a change of interpretation, we can regard M as a real $4m$-dimensional manifold. Then the real part ω_R of $\tilde{\omega}$ defines a symplectic structure on M in a natural way, while the imaginary part ω_I defines a co-symplectic structure. More particularly, consider a local chart of canonical coordinates for the symplectic structure ω. In terms of these coordinates,

$$\omega = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \tag{26}$$

where I is the $m \times m$ identity matrix. If we now set

$$\xi = (u + iv, u' + iv')$$

$$\eta = (p + iq, p' + iq')$$

we get

$$\omega_I(((u, v), (u', v')), ((p, q), (p', q')))$$
$$= ((u, v), (u', v'))\begin{pmatrix} 0 & 0 & 0 & I_m \\ 0 & 0 & -I_m & 0 \\ 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \\ \begin{pmatrix} p' \\ q' \end{pmatrix} \end{pmatrix} \tag{27}$$

It is clear that the bilinear form ω_I has signature zero, and hence is co-symplectic.

6. TWO-DIMENSIONAL ASSOCIATED STRUCTURES

Let M be a two-dimensional manifold. Then a symplectic structure on M is defined by any nonsingular 2-form

$$\omega(x) = \begin{pmatrix} 0 & a(x) \\ -a(x) & 0 \end{pmatrix}$$

where $a(x)$ is a 1-density. A co-symplectic structure (M, g) associated with (M, ω) has the following form in canonical coordinates:

$$(g_{ij}(x)) = \begin{pmatrix} G(x) & 1 \\ 1 & 0 \end{pmatrix} \tag{28}$$

We note in passing that the Riemannian hyperbolic tensor (28) provides a two-dimensional model of relativity that is of no small interest in its own right.

We now evaluate the components Γ^i_{jk} of the connection for g_{ij}, and its curvature, in a local coordinate system. We have

$$\begin{aligned} g &= -1 \\ g^{11} &= 0 \\ g^{12} &= g^{21} = 1 \\ g^{22} &= -G \end{aligned}$$

so that

$$\begin{aligned} \Gamma^1_{11} &= -\tfrac{1}{2}\partial_2 G \\ \Gamma^1_{12} &= \Gamma^1_{21} = \Gamma^1_{22} = 0 \end{aligned}$$

and

$$\begin{aligned} \Gamma^2_{11} &= \tfrac{1}{2}(\partial_1 G + G\partial_2 G) \\ \Gamma^2_{12} &= \Gamma^2_{21} = \tfrac{1}{2}\partial_2 G \\ \Gamma^2_{22} &= 0 \end{aligned}$$

The scalar curvature of g_{ij} is thus

$$K = \tfrac{1}{2}\partial_2^2 G \tag{29}$$

where $\partial_i = \partial/\partial x^i$. Note that, locally at least, any arbitrary function can be taken as the scalar curvature of some g_{ij}.

If g_{ij} defines a co-symplectic structure cotangentially associated with (M, ω), then G is a linear function of x^2 and its scalar curvature vanishes. In particular, let X be a one-dimensional manifold and (U, q) a local chart on X. Then in the natural coordinates $(U^*, \{q, p\})$ induced on T^*X we get

$$\sigma(q, p) = \begin{pmatrix} S(q)p & 1 \\ 1 & 0 \end{pmatrix}$$

and consequently the scalar curvature of σ vanishes.

7. ASSOCIATED STRUCTURES IN GENERAL

We shall now evaluate the Christoffel symbols of g given by (19). Let $1 \leq \alpha, \beta, \gamma \leq n$ and $n+1 \leq i, j, k \leq 2n$. Then

$$g_{\alpha\beta} = G_{\alpha\beta}$$

$$g_{\alpha i} = g_{i\alpha} = \delta_{i,\alpha+n}$$

$$g_{ij} = 0$$

and

$$g^{\alpha\beta} = 0$$

$$g^{\alpha i} = g^{i\alpha} = \delta^{i,\alpha+n}$$

$$g^{\alpha+n,\beta+n} = -G_{\alpha\beta}$$

Hence,

$$\Gamma^{\alpha}{}_{\beta\gamma} = -\tfrac{1}{2}\partial_{\alpha+n} G_{\beta\gamma}$$

$$\Gamma^{\alpha}{}_{\beta i} = \Gamma^{\alpha}{}_{i\beta} = \Gamma^{\alpha}{}_{ij} = 0$$

$$\Gamma^{i}{}_{\alpha\beta} = \tfrac{1}{2}(\partial_{\alpha} G_{\beta,i-n} + \partial_{\beta} G_{\alpha,i-n} - \partial_{i-n} G_{\alpha b}) + \tfrac{1}{2}\sum_{j=n+1}^{2n} G_{i-n,j-n}\partial_j G_{\alpha\beta} \tag{30}$$

$$\Gamma^{i}{}_{\alpha j} = \Gamma^{i}{}_{j\alpha} = \tfrac{1}{2}\partial_j G_{i-n,\alpha}$$

$$\Gamma^{i}{}_{jk} = 0$$

Suppose now that $G_{\alpha\beta} = G_{\alpha\beta,\gamma}(p)q^{\gamma}$. Then

$$\Gamma^{\alpha}{}_{\beta\gamma} = -\tfrac{1}{2}(\partial_{\alpha+n} G_{\beta\gamma,\omega})q^{\omega}$$

$$\Gamma^{\alpha}{}_{i\beta} = \Gamma^{\alpha}{}_{\beta i} = \Gamma^{\alpha}{}_{ij} = 0$$

$$\Gamma^{i}{}_{\alpha\beta} = \tfrac{1}{2}\sum_{s=n+1}^{2n} G_{i-n,s-n,\gamma}(\partial_s G_{\alpha\beta,\rho})q^{\gamma}q^{\rho} + \tfrac{1}{2}(G_{\alpha,i-n,\beta} + G_{\beta,i-n,\alpha} - G_{\alpha\beta,i-n}) \tag{31}$$

$$\Gamma^{i}{}_{\alpha j} = \Gamma^{i}{}_{j\alpha} = \tfrac{1}{2}(\partial_j G_{\alpha\beta,\gamma})q^{\gamma}$$

$$\Gamma^{i}{}_{jk} = 0$$

Due to the simple form of the inverse matrix to (g_{ab}), we obtain polynomial formulas for the curvature tensor. We have the following cases:

$$R_{\alpha\beta\gamma}{}^{\delta} = -\tfrac{1}{2}\partial^2_{\alpha,\delta+n} G_{\beta\gamma} + \tfrac{1}{2}\partial^2_{\beta,\delta+n} G_{\alpha\gamma} + \tfrac{1}{4}\sum_{\omega=1}^{n} [(\partial_{\delta+n} G_{\alpha\omega})(\partial_{\omega+n} G_{\beta\gamma}) - (\partial_{\delta+n} G_{\beta\omega})(\partial_{\omega+n} G_{\alpha\gamma})] \tag{32}$$

$$R_{i\alpha\omega}{}^{\gamma} = -R_{\alpha i\omega}{}^{\gamma} = -\tfrac{1}{2}\partial^2_{i,\gamma+n} G_{\alpha\omega} \tag{33}$$

$$R_{\alpha\beta k}{}^{\gamma} = 0 \tag{34}$$

$$\begin{aligned} R_{\alpha\beta\gamma}{}^{l} = {} & \tfrac{1}{4}\sum_{\rho=1}^{n} [(\partial_\alpha G_{l-n,\rho})(\partial_{\rho+n} G_{\beta\gamma}) - (\partial_\beta G_{l-n,\rho})(\partial_{\rho+n} G_{\alpha\gamma})] \\ & + \tfrac{1}{2}\sum_{s=n+1}^{2n} G_{l-n,s-n}(\partial^2_{\alpha,s} G_{\beta\gamma} - \partial^2_{\beta,s} G_{\alpha\gamma}) \\ & + \tfrac{1}{4}\sum_{s=n+1}^{2n} \partial_s G_{\alpha,l-n}(\partial_\beta G_{\gamma,s-n} + \partial_\gamma G_{\beta,s-n} - \partial_{s-n} G_{\beta\gamma} \\ & + \sum_{j=n+1}^{2n} G_{s-n,j-n}\partial_j G_{\beta\gamma}) \\ & - \tfrac{1}{4}\sum_{s=n+1}^{2n} \partial_s G_{\beta,l-n}(\partial_\alpha G_{\gamma,s-n} + \partial_\gamma G_{\alpha,s-n} - \partial_{s-n} G_{\alpha\gamma} \\ & + \sum_{j=n+1}^{2n} G_{s-n,j-n}\partial_j G_{\alpha\gamma}) \\ & - \tfrac{1}{4}\sum_{\omega=1}^{n}\left(\sum_{s=n+1}^{2n} G_{l-n,s-n}\partial_s G_{\alpha\omega} + \partial_\omega G_{\alpha,l-n} - \partial_{l-n} G_{\alpha\omega}\right)\partial_{\omega+n} G_{\beta\gamma} \\ & + \tfrac{1}{4}\sum_{\omega=1}^{n}\left(\sum_{s=n+1}^{2n} G_{l-n,s-n}\partial_s G_{\beta\omega} + \partial_\omega G_{\beta,l-n} - \partial_{l-n} G_{\beta\omega}\right)\partial_{\omega+n} G_{\alpha\gamma} \\ & + \tfrac{1}{2}(\partial^2_{\alpha,\gamma} G_{\beta,l-n} - \partial^2_{\beta,\gamma} G_{\alpha,l-n} + \partial^2_{\beta,l-n} G_{\alpha\gamma} - \partial^2_{\alpha,l-n} G_{\beta\gamma}) \end{aligned} \tag{35}$$

$$R_{ij\alpha}{}^{\beta} = R_{i\alpha j}{}^{\beta} = R_{\alpha ij}{}^{\beta} = 0 \tag{36}$$

$$\begin{aligned} R_{i\alpha\beta}{}^{l} = -R_{\alpha i\beta}{}^{l} = {} & \tfrac{1}{2}(\partial^2_{i\beta} G_{\alpha,l-n} - \partial^2_{i,l-n} G_{\alpha\beta}) + \tfrac{1}{2}\sum_{s=n+1}^{2n} G_{l-n,s-n}\partial^2_{i,s} G_{\alpha\beta} \\ & - \tfrac{1}{4}\sum_{\gamma=1}^{n} (\partial_i G_{\gamma,l-n})(\partial_{\gamma+n} G_{\alpha\beta}) - \tfrac{1}{4}\sum_{s=n+1}^{2n} (\partial_s G_{\alpha,l-n})(\partial_i G_{\beta,s-n}) \end{aligned} \tag{37}$$

$$\begin{aligned} R_{\alpha\beta k}{}^{l} = {} & \tfrac{1}{2}\partial^2_{\alpha,k} G_{\beta,l-n} - \tfrac{1}{2}\partial^2_{\beta,k} G_{\alpha,l-n} + \tfrac{1}{4}\sum_{s=n+1}^{2n} (\partial_s G_{\alpha,l-n})(\partial_k G_{\beta,s-n}) \\ & - \tfrac{1}{4}\sum_{s=n+1}^{2n} (\partial_s G_{\beta,l-n})(\partial_k G_{\alpha,s-n}) \end{aligned} \tag{38}$$

$$R_{i\alpha k}{}^{l} = -R_{\alpha ik}{}^{l} = \tfrac{1}{2}\partial^2_{i,k} G_{\alpha,l-n} \tag{39}$$

$$R_{ij\alpha}{}^{k} = R_{ijk}{}^{\alpha} = R_{ijk}{}^{l} = 0 \tag{40}$$

It is evident from (30) and (32)-(40) that for dim $X \geq 2$, the co-symplectic Riemann tensor on T^*X in general has nonzero curvature and Ricci tensors.

Next we evaluate the covariant derivative of the symplectic tensor in canonical coordinates. We have

$$\nabla_a \omega_{bc} = 0 \tag{41}$$

if at least one of the indices a, b, c belongs to $n+1, \ldots, 2n$. Also

$$\nabla_\alpha \omega_{\beta\gamma} = \tfrac{1}{2} \sum_{j=n+1}^{2n} (G_{\gamma,j-n} \partial_j G_{\alpha\beta} - G_{\beta,j-n} \partial_j G_{\alpha\gamma}) + \partial_\beta G_{\alpha\gamma} - \partial_\gamma G_{\alpha\beta} \tag{42}$$

We thus have

Proposition 2. The symplectic structure (M, ω) is covariantly constant with respect to the Christoffel connection of a co-symplectic structure (M, σ) associated with (M, ω) if and only if

$$\tfrac{1}{2} \sum_{j=n+1}^{2n} (G_{\gamma,j-n} \partial_j G_{\alpha\beta} - G_{\beta,j-n} \partial_j G_{\alpha\gamma}) + \partial_\beta G_{\alpha\gamma} - \partial_\gamma G_{\alpha\beta} = 0 \tag{43}$$

We now consider the equations of geodesics. We have

$$0 = \frac{d^2 x^\alpha}{dt^2} - \sum_{\substack{\beta,\gamma \\ \beta \neq \gamma}} \partial_{\alpha+n} G_{\beta\gamma} \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} - \tfrac{1}{2} \sum_\beta \partial_{\alpha+n} G_{\beta\beta} \left(\frac{dx^\beta}{dt} \right)^2 \tag{44}$$

$$\begin{aligned} 0 = \frac{d^2 x^i}{dt} &+ \sum_{j,\alpha} \partial_j G_{\alpha,i-n} \frac{dx^j}{dt} \frac{dx^\alpha}{dt} \\ &+ \tfrac{1}{2} \sum_{\substack{\alpha,\beta \\ \alpha \neq \beta}} \sum_s G_{i-n,s-n} \partial_s G_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \\ &+ \tfrac{1}{2} \sum_{\substack{\alpha,\beta \\ \alpha \neq \beta}} (\partial_\alpha G_{\beta,i-n} + \partial_\beta G_{\alpha,i-n} - \partial_{i-n} G_{\alpha\beta}) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \\ &+ \tfrac{1}{2} \sum_\alpha \sum_s G_{i-n,s-n} \partial_s G_{\alpha\alpha} \left(\frac{dx^\alpha}{dt} \right)^2 \\ &+ \sum_\alpha \left(\partial_\alpha G_{\alpha,i-n} - \tfrac{1}{2} \partial_{i-n} G_{\alpha\beta} \right) \left(\frac{dx^\alpha}{dt} \right)^2 \end{aligned} \tag{45}$$

Note in particular that the curves

$$x^\alpha = c^\alpha$$

$$x^i = m^i t + n^i$$

where $\alpha = 1, \ldots, n$ and $i = n+1, \ldots, 2n$, are geodesics. If we are working on the cotangent bundle, this observation yields the following results.

Proposition 3. Let (T^*X, σ) be a co-symplectic structure associated with the standard symplectic structure on T^*X. Then the submanifold $X \subset T^*X$ is a geodesic submanifold, i.e., for any $x, y \in X$, x and y being sufficiently close to each other, there is a geodesic of (T^*X, σ) in X which joins the points x and y.

8. CO-SYMPLECTIC STRUCTURE AND POLARIZATION

Let (M, σ) be a co-symplectic structure on a manifold M. Consider a coordinate system (U, x) on M in which σ is given by (19), that is,

$$\sigma = \begin{pmatrix} G & I_n \\ I_n & 0 \end{pmatrix}$$

If $x = (x^1, \ldots, x^{2n})$ and $1 \le \alpha \le n$, $n+1 \le i \le 2n$, then the submanifolds $V_c(x)$, with $c = (c^1, \ldots, c^n)$, given by

$$x^\alpha = c^\alpha \tag{46}$$

define a local geodesic foliation on U. The connection induced by ∇ on V_c is flat. These are straightforward conclusions from equations (30), (44), and (45).

We shall now demonstrate that if (M, σ) is associated with a symplectic structure (M, ω), then this local geodesic foliation extends to a global geodesic foliation. Suppose (U, x) and (V, y) are two overlapping canonical charts. Then the transition function preserves both ω and σ. Construct the mixed tensor

$$T = \omega^{-1}\sigma = \begin{pmatrix} I_n & 0 \\ -G & -I_n \end{pmatrix} \tag{47}$$

The tangent spaces $T_x V_c$ are then clearly the eigensubspaces of T corresponding to the eigenvalue -1. Since T is preserved under the local coordinate transformation $x \to y$, which is guaranteed by the fact that (U, x) and (V, y) are canonical, its eigensubspaces will also be preserved. Hence, it follows that the local geodesic foliation (46) extends to a global foliation.

The tensor T has a second n-dimensional eigensubspace $V'_c(x)$ corresponding to the eigenvalue $+1$. This subspace can be parametrized by

$$V'_c(x) = \left\{ \xi \in T_x M;\ \xi = \begin{pmatrix} I_n \\ -\frac{1}{2}G \end{pmatrix} \eta,\ \eta \in \Re^n \right\} \tag{48}$$

Since G is symmetric (Frescura and Lubczonok, 1990), the subspace $V_c'(x)$ is a Lagrangian subspace for every $x \in U$. An easy calculation shows that $V_c'(x)$ is also a co-Lagrangian subspace, i.e., a maximal isotropic subspace of σ. The distribution $x \to V_c'(x)$ is not always involutive, even in the case when $M = T^*X$.

Suppose now that ω is covariantly constant with respect to the connection ∇, so that equation (43) is satisfied. Then, clearly, the tensor T, and thus also the distributions $\{V_c(x)\}$ and $\{V_c'(x)\}$, are covariantly constant. This gives the following result.

Proposition 4. Let (M, ω) be a symplectic structure on a manifold M, and (M, σ) a co-symplectic structure associated with (M, ω). The distribution $x \to V_c(x)$ is a geodesic, Lagrangian, and co-Lagrangian foliation on M. The metric connection defined by σ, restricted to the leaves of this foliation, is flat. The complementary distribution $x \to V_c'(x)$ is Lagrangian and co-Lagrangian. If ω is covariantly constant with respect to the metric connection, then the distributions $V_c(x)$ and $V_c'(x)$ are parallel.

We shall now consider a special case that is of some importance for geometric quantization. Suppose the foliation $V_c(x)$, where $x \in M$, admits a section Q that is both Lagrangian and co-Lagrangian.[4] Then, by a straightforward application of Proposition 4.4.1 of Woodhouse (1980) to our polarization, we obtain the following results.

Proposition 5. Let $P = \{V_c(x)\}$, $x \in M$, be the polarization, and let $Q \subset M$ be a Lagrangian and a co-Lagrangian manifold which is also a section of P. Then there is a natural canonical diffeomorphism $\rho: U \to M$, where U is a neighborhood of the zero section in T^*Q endowed with its natural symplectic structure, such that:

1. $\rho^{-1}(Q)$ is the zero section in $U \subset T^*Q$
2. $\rho^*(P)$ is the vertical polarization of U
3. $\rho_*(\sigma)$ defines a co-symplectic structure on $U \subset T^*Q$ associated with the natural symplectic structure on T^*Q.

Moreover, if the leaves of P are geodesically complete, then $U = T^*Q$, and ρ identifies (M, ω) with the natural symplectic structure on T^*Q.

The proof of this proposition follows from the fact that P is convex.

9. CONCLUSION

We have introduced a co-symplectic structure on the cotangent bundle T^*X of an arbitrary paracompact manifold X, and defined the co-symplectic

[4] Q is a section of a foliation if it intersects each leaf of the foliation in exactly one point.

structure associated with a given symplectic manifold (M, ω). In general, ω is not covariantly constant with respect to the co-symplectic connection for arbitrary associated co-symplectic structures (M, σ). The necessary and sufficient conditions for this to be the case are given by equation (43). In the case $M = T^*X$ for some arbitrary manifold X, X considered as a subspace of T^*X is a geodesic submanifold.

The pair (M, ω) and (M, σ) define a global Lagrangian and co-Lagrangian geodesic foliation on M. This foliation and its complementary distribution produce the polarization of the symplectic manifold (M, ω) that is so fundamental to geometric quantization. The simultaneous presence of a Riemannian structure, in the form of the associated co-symplectic geometry, may prove to be important in putting together the bosonic and the fermionic aspects of quantum theory.

REFERENCES

Abraham, R., and Marsden, J. E. (1978). *Foundations of Mechanics*, Benjamin/Cummings, Reading, Massachusetts.

Berezin, F. A. (1966). *The Method of Second Quantisation*, Academic Press, New York.

Berezin, F. A. (1987). *Introduction to Superanalysis*, Reidel, Dordrecht.

Berezin, F. A., and Marinov, M. (1977). *Annals of Physics*, **104**, 336.

Cariñena, J. F., and Ibort, L. A. (1984). *Letters in Mathematical Physics*, **8**, 359-365.

Frescura, F. A. M., and Lubczonok, G. (1990). *International Journal of Theoretical Physics*, **29**, 57-73.

Gomis, J., Novell, M., Poch, A., and Rafanelli, K. (1985). *Physical Review*, **32D**, 1985-92.

Guillemin, V., and Sternberg, S. (1977). *Geometric Asymptotics* (Mathematical Surveys, Number 14), American Mathematical Society, Providence, Rhode Island.

Guillemin, V., and Sternberg, S. (1984). *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge.

Hess, H. (1980). Connections on symplectic manifolds and geometric quantisation, in *Differential Geometric Methods of Mathematical Physics* (Springer Lecture Notes in Mathematics, Vol. 836), Springer-Verlag, New York.

Liebermann, P., and Marle, C. (1987). *Symplectic Geometry and Analytical Mechanics*, Reidel, Dordrecht.

Porteous, I. R. (1969). *Topological Geometry*, Van Nostrand Reinhold, London.

Sherry, G. (1989). *Foundations of Physics*, **19**, 733.

Woodhouse, N. (1980). *Geometric Quantisation* (Oxford Mathematical Monographs), Clarendon Press, Oxford.